9

Maxwell's equations and electromagnetic waves

Overview In the course of our study of electromagnetism, we have been gradually putting together the pieces of the puzzle of *Maxwell's equations.* In this chapter we find the final piece, known as the *displacement current.* We do this by exposing a contradiction in our present theory and then resolving it. Once we write down the full set of Maxwell's equations, we quickly discover that in vacuum they lead to *wave* solutions with a set of specific properties. These waves are light waves, and unlike other waves you are familiar with, they require no medium to support their propagation. *Traveling* electromagnetic waves carry energy, and more generally the *Poynting vector* describes the energy flow in an arbitrary electromagnetic field. *Standing* electromagnetic waves, which are the superposition of traveling waves, carry no net energy. By examining how the electric and magnetic fields transform between frames, we find that a light wave in one frame looks like a light wave in any other frame.

9.1 "Something is missing"

Let us review the relations between charges and fields. As we learned in Chapter 2, a statement equivalent to Coulomb's law is the differential form of Gauss's law,

$$\boxed{\operatorname{div}\mathbf{E} = \frac{\rho}{\epsilon_0}} \tag{9.1}$$

connecting the electric charge density ρ and the electric field $\mathbf{E}$. This holds for moving charges as well as stationary charges. That is, ρ can be a function of time as well as position. As we emphasized in Chapter 5, the fact that Eq. (9.1) holds for moving charges is consistent with *charge invariance:* no matter how an isolated charged particle may be moving, its charge, as measured by the integral of $\mathbf{E}$ over a surface surrounding it, appears the same in every frame of reference.

Electric charge in motion is electric current. Because charge is never created or destroyed, the charge density ρ and the current density $\mathbf{J}$ always satisfy the condition

$$\boxed{\operatorname{div}\mathbf{J} = -\frac{\partial\rho}{\partial t}} \tag{9.2}$$

We first wrote down this "equation of continuity" as Eq. (4.10).

If the current density $\mathbf{J}$ is constant in time, we call it a *stationary current distribution*. The magnetic field of such a current satisfies the equation

$$\operatorname{curl}\mathbf{B} = \mu_0\mathbf{J} \qquad \text{(stationary current distribution)}. \tag{9.3}$$

We worked with this relation in Chapter 6.

Now we are interested in charge distributions and fields that are changing in time. Suppose we have a charge distribution $\rho(x, y, z, t)$ with $\partial\rho/\partial t \neq 0$. For instance, we might have a capacitor that is discharging through a resistor. According to Eq. (9.2), $\partial\rho/\partial t \neq 0$ implies

$$\operatorname{div}\mathbf{J} \neq 0. \tag{9.4}$$

But according to Eq. (9.3), since the divergence of the curl of *any* vector function is identically zero (see Exercise 2.78),

$$\operatorname{div}\mathbf{J} = \frac{1}{\mu_0}\operatorname{div}(\operatorname{curl}\mathbf{B}) = 0. \tag{9.5}$$

The contradiction shows that Eq. (9.3) *cannot be correct* for a system in which the charge density is varying in time. Of course, no one claimed it was; a stationary current distribution, for which Eq. (9.3) *does* hold, is one in which not even the current density $\mathbf{J}$, let alone the charge density ρ, is time-dependent.

The problem can be posed in somewhat different terms by considering the line integral of magnetic field around the wire that carries charge away from the capacitor plate in Fig. 9.1. According to Stokes' theorem,

$$\int_C \mathbf{B}\cdot d\mathbf{l} = \int_S \operatorname{curl}\mathbf{B}\cdot d\mathbf{a}. \tag{9.6}$$

The surface S passes right through the conductor in which a current I is flowing. Inside this conductor, $\operatorname{curl}\mathbf{B}$ has a finite value, namely $\mu_0\mathbf{J}$, and the integral on the right comes out equal to $\mu_0 I$. That is to say, if the curve C is close to the wire and well away from the capacitor gap, the

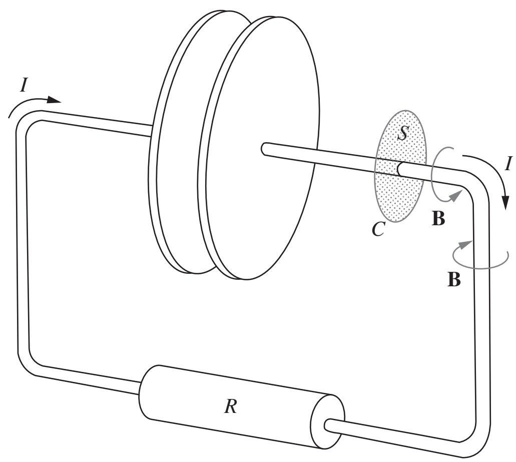


Figure 9.1.
Having been charged with the right-hand plate positive, the capacitor is being discharged through the resistor. There is a magnetic field $\mathbf{B}$ around the wire. The integral of curl $\mathbf{B}$, over the surface S that passes through the wire, has the value $\mu_0 I$.

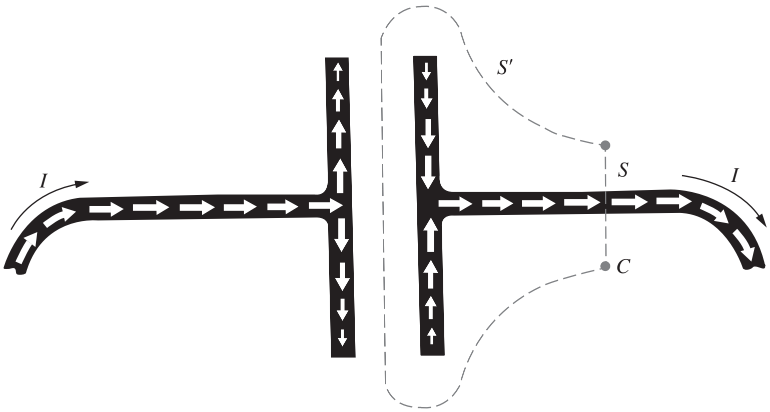


Figure 9.2.
The white arrows show the current flow in the conductors. The surface S', which like S has the curve C for its edge, has no current passing through it.

magnetic field there is not different from the field around any wire carrying the same current. Now, the surface S' in Fig. 9.2 is also a surface spanning C, and has an equally good claim to be used in the statement of Stokes' theorem, Eq. (9.6). Through this surface, however, there flows *no current at all!* Nevertheless, curl $\mathbf{B}$ cannot be zero over all of S' without violating Stokes' theorem. Therefore, on S', curl $\mathbf{B}$ must depend on something other than the current density $\mathbf{J}$.

We can only conclude that Eq. (9.3) has to be replaced by some other relation, in the more general situation of changing charge distributions. Let's write instead

$$\text{curl}\,\mathbf{B} = \mu_0\mathbf{J} + (?) \tag{9.7}$$

and see if we can discover what (?) must be.

Another line of thought suggests the answer. Remember that the Lorentz-transformation laws of the electromagnetic field, Eq. (6.73), are symmetrical in $\mathbf{E}$ and $c\mathbf{B}$. Now, in Faraday's induction phenomenon, a *changing magnetic field* is accompanied by an *electric field,* in a manner described by Eq. (7.31):

$$\boxed{\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{9.8}$$

This is a local relation connecting the electric and magnetic fields in empty space – charges are not directly involved. If symmetry with respect to $\mathbf{E}$ and $c\mathbf{B}$ is to prevail, we must expect that a *changing electric field* can give rise to a *magnetic field.* There ought to be an induction phenomenon described by an equation like Eq. (9.8), but with the roles of $\mathbf{E}$ and $c\mathbf{B}$ switched. Writing Eq. (9.8) as $\text{curl}\,\mathbf{E} = -(1/c)\partial(c\mathbf{B})/\partial t$ and then reversing the roles of $\mathbf{E}$ and $c\mathbf{B}$, we obtain $\text{curl}\,(c\mathbf{B}) = -(1/c)\partial\mathbf{E}/\partial t \Longrightarrow \text{curl}\,\mathbf{B} = -(1/c^2)\partial\mathbf{E}/\partial t$. It will turn out that we need to change the sign in order for Eq. (9.13) below to work out correctly, but that is all:

$$\operatorname{curl} \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \implies \operatorname{curl} \mathbf{B} = \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{9.9}$$

where we have used the relation $c^2 = 1/\mu_0\epsilon_0$ from Eq. (6.8). The second of the expressions in Eq. (9.9) is the standard way of writing the relation in SI units.

This provides the missing term that is called for in Eq. (9.7). To try it out, write

$$\boxed{\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}} \tag{9.10}$$

and take the divergence of both sides:

$$\operatorname{div}(\operatorname{curl} \mathbf{B}) = \operatorname{div}(\mu_0 \mathbf{J}) + \operatorname{div}\left(\mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right). \tag{9.11}$$

The left side is necessarily zero, as already remarked. In the second term on the right we can interchange the order of differentiation with respect to space coordinates and time. Thus

$$\operatorname{div}\left(\mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) = \mu_0\epsilon_0 \frac{\partial}{\partial t}(\operatorname{div} \mathbf{E}) = \mu_0\epsilon_0 \frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right) = \mu_0 \frac{\partial \rho}{\partial t}, \tag{9.12}$$

by Eq. (9.1). The right-hand side of Eq. (9.11) now becomes

$$\mu_0 \operatorname{div} \mathbf{J} + \mu_0 \frac{\partial \rho}{\partial t}, \tag{9.13}$$

which is zero by virtue of the continuity condition, Eq. (9.2).

The new term resolves the difficulty raised in Fig. 9.2. As charge flows out of the capacitor, the electric field, which at any instant has the configuration in Fig. 9.3, *diminishes* in intensity. In this case, $\partial \mathbf{E}/\partial t$ points opposite to $\mathbf{E}$. The vector function $\mu_0\epsilon_0(\partial \mathbf{E}/\partial t)$ is represented by the black arrows in Fig. 9.4. With $\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0(\partial \mathbf{E}/\partial t)$, the integral of $\operatorname{curl} \mathbf{B}$ over S' now has the same value as it does over S. On S' the second term contributes everything; on S the first term, the term with $\mathbf{J}$, is practically all that counts.

9.2 The displacement current

Observe that the vector field $\mu_0\epsilon_0(\partial \mathbf{E}/\partial t)$ appears to form a *continuation* of the conduction current distribution. Maxwell called it the *displacement current,* and the name has stuck although it no longer seems very appropriate. To be precise, we can define a *displacement current density* $\mathbf{J}_\mathrm{d}$, to be distinguished from the conduction current density $\mathbf{J}$, by writing Eq. (9.10) this way:

$$\operatorname{curl} \mathbf{B} = \mu_0(\mathbf{J} + \mathbf{J}_\mathrm{d}), \tag{9.14}$$

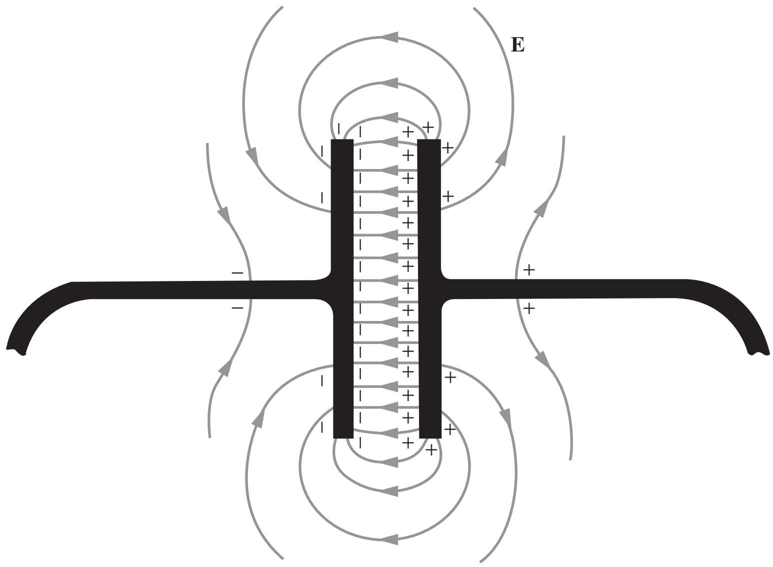


Figure 9.3.
The electric field at a particular instant. The magnitude of $\mathbf{E}$ is decreasing everywhere as time goes on.

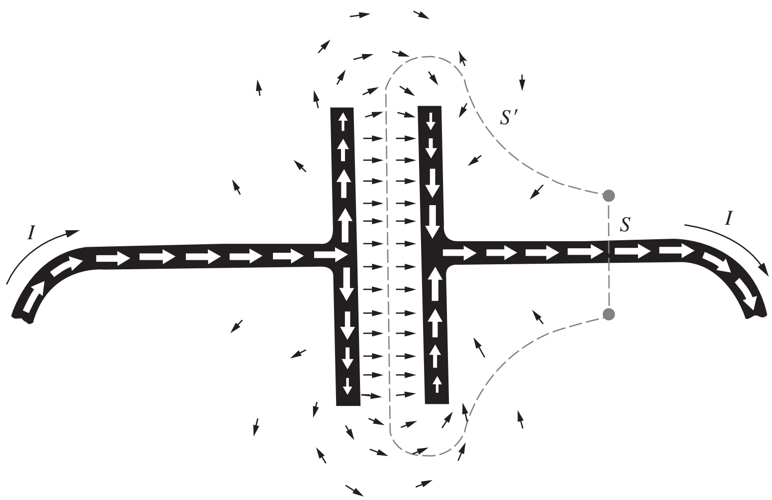


Figure 9.4.
The conduction current (white arrows) and the displacement current (black arrows).

and defining

$$\mathbf{J}_\mathrm{d} \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} . \tag{9.15}$$

We needed the new term to make the relation between current and magnetic field consistent with the continuity equation, in the case of conduction currents changing in time. If it belongs there, it implies the existence of a new induction effect in which a changing electric field is accompanied by a magnetic field. If the effect is real, why didn't

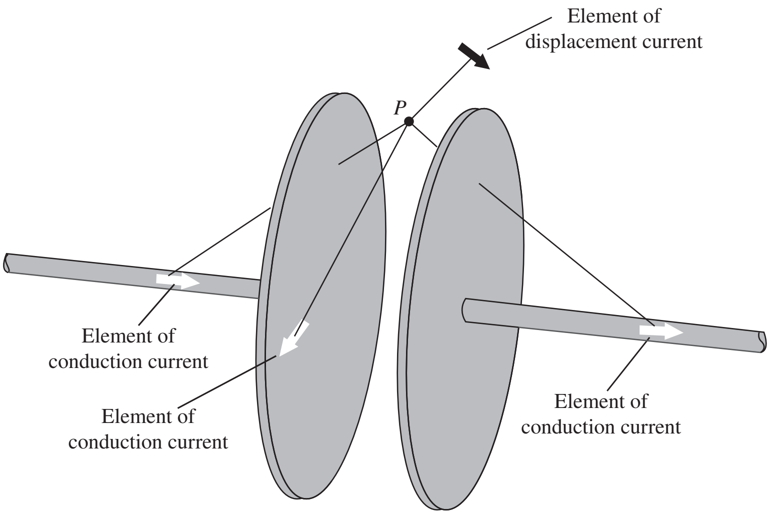


Figure 9.5.
In the case of slowly varying fields, the total contribution to the magnetic field at any point, from all displacement currents, is zero. The magnetic field at P can be calculated by the Biot–Savart formula applied to conduction current elements only.

Faraday discover it? For one thing, he wasn't looking for it, but there is a more fundamental reason why experiments like Faraday's could not have revealed any new effects attributable to the last term in Eq. (9.10). In any apparatus in which there are changing electric fields, there are present, at the same time, conduction currents, charges in motion. The magnetic field **B**, everywhere around the apparatus, is just about what you would expect those conduction currents to produce. In fact, it is almost exactly the field you would calculate if, ignoring the fact that the circuits may not be continuous, you use the Biot–Savart formula, Eq. (6.49), to find the contribution of each conduction current element to the field at some point in space.

Consider, for example, the point P in the space between our discharging capacitor plates, Fig. 9.5. Each element of conduction current, in the wires and on the surface of the plates, contributes to the field at P, according to the Biot–Savart formula. Must we include also the elements of displacement current density $\mathbf{J}_\mathrm{d}$? The answer is rather surprising. We *may* include $\mathbf{J}_\mathrm{d}$, but if we are careful to include the *entire* displacement current distribution, its net effect will be *zero* for relatively slowly varying fields.

To see why this is so, note that the vector function $\mathbf{J}_\mathrm{d}$, indicated by the black arrows in Fig. 9.4, has the same form as the electric field **E** in Fig. 9.3. This electric field is practically an electrostatic field, except that it is slowly dying away. We expect therefore that its curl is practically zero, which would imply that curl $\mathbf{J}_\mathrm{d}$ must be practically zero. More precisely, we have curl $\mathbf{E} = -\partial \mathbf{B}/\partial t$, and with the displacement current $\mathbf{J}_\mathrm{d} = \epsilon_0(\partial \mathbf{E}/\partial t)$, we get, by interchanging the order of differentiation,

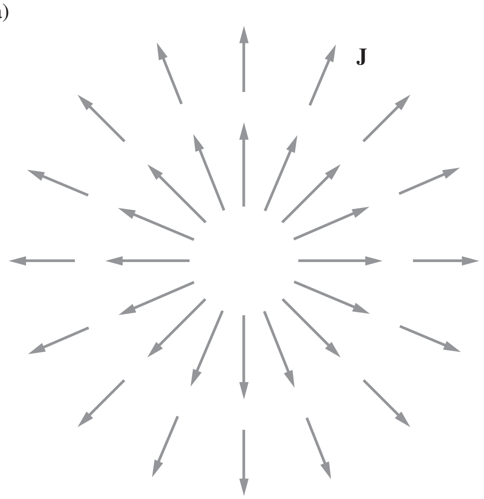
$$\text{curl}\, \mathbf{J}_\mathrm{d} = \epsilon_0 \,\text{curl} \left(\frac{\partial \mathbf{E}}{\partial t}\right) = \epsilon_0 \frac{\partial}{\partial t}(\text{curl}\, \mathbf{E}) = -\epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}\,. \tag{9.16}$$

This will be negligible for sufficiently slow changes in field. We may call a slowly changing field *quasi-static*. Now, if $\mathbf{J}_d$ is a vector field without any curl, it can be made up, in the same way that the electrostatic field can be made of the radial fields of point charges, by superposing radial currents flowing outward from point sources or in toward point "sinks" (Fig. 9.6). But the magnetic field of any *radial,* symmetrical current distribution, calculated via Biot–Savart, is zero. To understand why, consider the radial line through a given location. At this location, the Biot–Savart contributions from a pair of points symmetrically located with respect to this line are equal and opposite, as you can verify. The contributions therefore cancel in pairs, yielding zero field at the given location.

In the quasi-static field, then, the conduction currents alone are the only *sources* needed to account for the magnetic field. In other words, if Faraday had arranged something like Fig. 9.5, and had been able to measure the magnetic field at P, by using a compass needle, say, he would not have been surprised. He would not have needed to invent a displacement current to explain it.

To see this new induction effect, we need rapidly changing fields. In fact, we need changes to occur in the time it takes light to cross the apparatus. That is why the direct demonstration had to wait for Hertz, whose experiment came roughly 25 years after the law itself had been worked out by Maxwell.

(a)



(b)

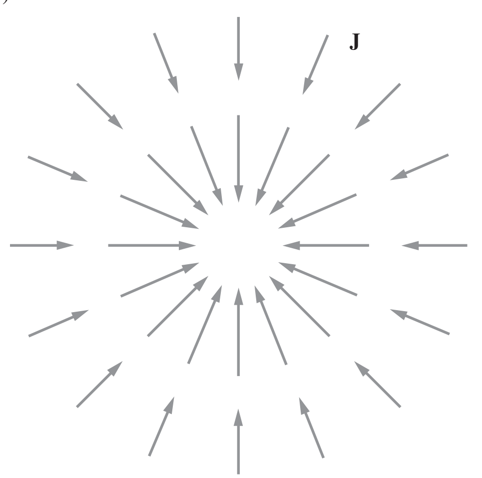


Figure 9.6.
Showing what is meant by a radial current distribution. The current density $\mathbf{J}$ for the point source in (a), or for the point "sink" in (b), is like the electric field of a point charge. Any current distribution with curl $\mathbf{J} = 0$ could be made by superposing such sources and sinks, and must therefore have zero magnetic field.

9.3 Maxwell's equations

James Clerk Maxwell (1831–1879), after immersing himself in the accounts of Faraday's electrical researches, set out to formulate mathematically a theory of electricity and magnetism. Maxwell could not exploit relativity – that came 50 years later. The electrical constitution of matter was a mystery, the relation between light and electromagnetism unsuspected. Many of the arguments that we have used to make our next step seem obvious were unthinkable then. Nevertheless, as Maxwell's theory developed, the term we have been discussing, $\partial\mathbf{E}/\partial t$, appeared quite naturally in his formulation. He called it the displacement current. Maxwell was concerned with electric fields in solid matter as well as in vacuum, and when he talks about a displacement current he is often including some charge in motion, too. We'll clarify that point in Chapter 10 when we study electric fields in matter. Indeed, Maxwell thought of space itself as a medium, the "aether," so that even in the absence of solid matter, the displacement current was occurring *in* something. But never mind – his mathematical equations were perfectly clear and unambiguous, and his introduction of the displacement current was a *theoretical* discovery of the first rank.

Maxwell's description of the electromagnetic field was essentially complete. We have arrived by different routes at various pieces of it, which we shall now assemble in the form traditionally called *Maxwell's equations:*

$$\boxed{\begin{aligned} \operatorname{curl}\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} \\ \operatorname{curl}\mathbf{B} &= \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \mu_0\mathbf{J} \\ \operatorname{div}\mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \operatorname{div}\mathbf{B} &= 0 \end{aligned}} \tag{9.17}$$

These are written for fields in the presence of electric charge of density ρ and electric current, that is, charge in motion, of density $\mathbf{J}$.

The first equation is Faraday's *law of induction.* The second expresses the dependence of the magnetic field on the *displacement current* density, or rate of change of electric field, and on the *conduction current* density, or rate of motion of charge. (If $\partial\mathbf{E}/\partial t = 0$, this equation reduces to Ampère's law.) The third equation is equivalent to Coulomb's law; it is the differential form of Gauss's law. The fourth equation states that there are no sources of magnetic field *except* currents; that is, there are no magnetic monopoles. We shall have more to say about this aspect of Nature in Chapter 11.

Note that the lack of symmetry in these equations, with respect to $\mathbf{B}$ and $\mathbf{E}$ (or rather $c\mathbf{B}$ and $\mathbf{E}$; see Eq. (9.19)), is entirely due to the presence of electric charge and electric conduction current. In empty space, the terms with ρ and $\mathbf{J}$ are zero, and Maxwell's equations become

$$\boxed{\begin{aligned} \operatorname{curl}\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} &\qquad \operatorname{div}\mathbf{E} &= 0 \\ \operatorname{curl}\mathbf{B} &= \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} &\qquad \operatorname{div}\mathbf{B} &= 0 \end{aligned}} \tag{9.18}$$

Remembering that $\mu_0\epsilon_0 = 1/c^2$, we can write the two "induction" equations as

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial(c\mathbf{B})}{\partial t} \qquad \text{and} \qquad \operatorname{curl}(c\mathbf{B}) = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\,, \tag{9.19}$$

where the symmetry between $c\mathbf{B}$ and $\mathbf{E}$ is clear. This symmetry, after all, is what led us to the displacement current in the first place; see the paragraph preceding Eq. (9.9).

In Eq. (9.18) the displacement current term is all important. Its presence, along with its counterpart in the first equation, implies the possibility of *electromagnetic waves,* as we will see in Section 9.4. Recognizing this, Maxwell went on to develop with brilliant success an electromagnetic theory of light.

In Gaussian units Maxwell's equations look like this:

$$\boxed{\begin{aligned} \text{curl}\,\mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \text{curl}\,\mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \\ \text{div}\,\mathbf{E} &= 4\pi\rho \\ \text{div}\,\mathbf{B} &= 0 \end{aligned}} \tag{9.20}$$

And in empty space, with ρ and $\mathbf{J}$ equal to zero, these become

$$\boxed{\begin{aligned} \text{curl}\,\mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad & \text{div}\,\mathbf{E} &= 0 \\ \text{curl}\,\mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \qquad & \text{div}\,\mathbf{B} &= 0 \end{aligned}} \tag{9.21}$$

7.3 ■ MAXWELL'S EQUATIONS

7.3.1 ■ Electrodynamics Before Maxwell

So far, we have encountered the following laws, specifying the divergence and curl of electric and magnetic fields:

$$\begin{aligned}
&\text{(i)} && \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho && \text{(Gauss's law),}\\
&\text{(ii)} && \nabla \cdot \mathbf{B} = 0 && \text{(no name),}\\
&\text{(iii)} && \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && \text{(Faraday's law),}\\
&\text{(iv)} && \nabla \times \mathbf{B} = \mu_0 \mathbf{J} && \text{(Ampère's law).}
\end{aligned}$$

These equations represent the state of electromagnetic theory in the mid-nineteenth century, when Maxwell began his work. They were not written in so compact a form, in those days, but their physical content was familiar. Now, it happens that there is a fatal inconsistency in these formulas. It has to do with the old rule that divergence of curl is always zero. If you apply the divergence to number (iii), everything works out:

$$\nabla \cdot (\nabla \times \mathbf{E}) = \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}).$$

The left side is zero because divergence of curl is zero; the right side is zero by virtue of equation (ii). But when you do the same thing to number (iv), you get into trouble:

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0(\nabla \cdot \mathbf{J}); \tag{7.36}$$

the left side must be zero, but the right side, in general, is *not*. For *steady* currents, the divergence of **J** is zero, but when we go beyond magnetostatics Ampère's law cannot be right.

There's another way to see that Ampère's law is bound to fail for nonsteady currents. Suppose we're in the process of charging up a capacitor (Fig. 7.43). In integral form, Ampère's law reads

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.$$

I want to apply it to the Amperian loop shown in the diagram. How do I determine I_{enc}? Well, it's the total current passing through the loop, or, more precisely, the current piercing a surface that has the loop for its boundary. In this case, the *simplest* surface lies in the plane of the loop—the wire punctures this surface, so $I_{\text{enc}} = I$. Fine—but what if I draw instead the balloon-shaped surface in Fig. 7.43? *No* current passes through *this* surface, and I conclude that $I_{\text{enc}} = 0$! We never had this problem in magnetostatics because the conflict arises only when charge

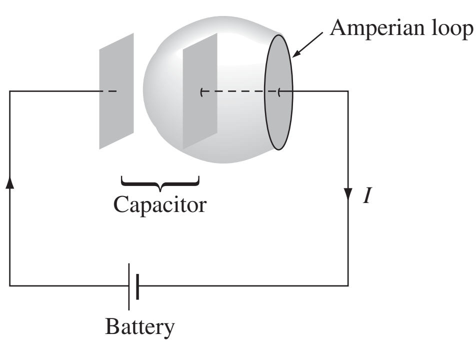


FIGURE 7.43

is piling up somewhere (in this case, on the capacitor plates). But for *nonsteady* currents (such as this one) "the current enclosed by the loop" is an ill-defined notion; it depends entirely on what surface you use. (If this seems pedantic to you—"obviously one should use the plane surface"—remember that the Amperian loop could be some contorted shape that doesn't even lie in a plane.)

Of course, we had no right to *expect* Ampère's law to hold outside of magnetostatics; after all, we derived it from the Biot-Savart law. However, in Maxwell's time there was no *experimental* reason to doubt that Ampère's law was of wider validity. The flaw was a purely theoretical one, and Maxwell fixed it by purely theoretical arguments.

7.3.2 ■ How Maxwell Fixed Ampère's Law

The problem is on the right side of Eq. 7.36, which *should be* zero, but *isn't*. Applying the continuity equation (5.29) and Gauss's law, the offending term can be rewritten:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right).$$

If we were to combine $\epsilon_0(\partial \mathbf{E}/\partial t)$ with $\mathbf{J}$, in Ampère's law, it would be just right to kill off the extra divergence:

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.} \tag{7.37}$$

(Maxwell himself had other reasons for wanting to add this quantity to Ampère's law. To him, the rescue of the continuity equation was a happy dividend rather than a primary motive. But today we recognize this argument as a far more compelling one than Maxwell's, which was based on a now-discredited model of the ether.)[20]

Such a modification changes nothing, as far as magneto*statics* is concerned: when $\mathbf{E}$ is constant, we still have $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. In fact, Maxwell's term is hard to detect in ordinary electromagnetic experiments, where it must compete for attention with $\mathbf{J}$—that's why Faraday and the others never discovered it in the laboratory. However, it plays a crucial role in the propagation of electromagnetic waves, as we'll see in Chapter 9.

Apart from curing the defect in Ampère's law, Maxwell's term has a certain aesthetic appeal: Just as a changing *magnetic* field induces an *electric* field (Faraday's law), so[21]

A changing electric field induces a magnetic field.

[20]For the history of this subject, see A. M. Bork, *Am. J. Phys.* **31**, 854 (1963).

[21]See footnote 8 (page 313) for commentary on the word "induce." The same issue arises here: Should a changing electric field be regarded as an independent source of magnetic field (along with current)? In a proximate sense it does function as a source, but since the electric field itself was produced by charges and currents, they alone are the "ultimate" sources of **E** and **B**. See S. E. Hill, *Phys. Teach.* **49**, 343 (2011); for a contrary view, see C. Savage, *Phys. Teach.* **50**, 226 (2012).

Of course, theoretical convenience and aesthetic consistency are only *suggestive*—there might, after all, be other ways to doctor up Ampère's law. The real confirmation of Maxwell's theory came in 1888 with Hertz's experiments on electromagnetic waves.

Maxwell called his extra term the **displacement current**:

$$\mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{7.38}$$

(It's a misleading name; $\epsilon_0(\partial \mathbf{E}/\partial t)$ has nothing to do with current, except that it adds to $\mathbf{J}$ in Ampère's law.) Let's see now how displacement current resolves the paradox of the charging capacitor (Fig. 7.43). If the capacitor plates are very close together (I didn't *draw* them that way, but the calculation is simpler if you assume this), then the electric field between them is

$$E = \frac{1}{\epsilon_0}\sigma = \frac{1}{\epsilon_0}\frac{Q}{A},$$

where Q is the charge on the plate and A is its area. Thus, between the plates

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I.$$

Now, Eq. 7.37 reads, in integral form,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \int \left(\frac{\partial \mathbf{E}}{\partial t}\right) \cdot d\mathbf{a}. \tag{7.39}$$

If we choose the *flat* surface, then $E = 0$ and $I_{\text{enc}} = I$. If, on the other hand, we use the balloon-shaped surface, then $I_{\text{enc}} = 0$, but $\int (\partial \mathbf{E}/\partial t) \cdot d\mathbf{a} = I/\epsilon_0$. So we get the same answer for either surface, though in the first case it comes from the conduction current, and in the second from the displacement current.

Example 7.14. Imagine two concentric metal spherical shells (Fig. 7.44).

The inner one (radius a) carries a charge $Q(t)$, and the outer one (radius b) an opposite charge $-Q(t)$. The space between them is filled with Ohmic material of conductivity σ, so a radial current flows:

$$\mathbf{J} = \sigma \mathbf{E} = \sigma \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}; \quad I = -\dot{Q} = \int \mathbf{J} \cdot d\mathbf{a} = \frac{\sigma Q}{\epsilon_0}.$$

This configuration is spherically symmetrical, so the magnetic field has to be zero (the only direction it could possibly point is radial, and $\nabla \cdot \mathbf{B} = 0 \Rightarrow \oint \mathbf{B} \cdot d\mathbf{a} = B(4\pi r^2) = 0$, so $\mathbf{B} = \mathbf{0}$). *What?* I thought currents produce magnetic fields! Isn't that what Biot-Savart and Ampère taught us? How can there be a $\mathbf{J}$ with no accompanying $\mathbf{B}$?

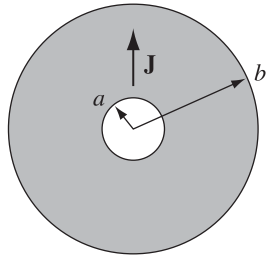


FIGURE 7.44

Solution

This is not a static configuration: Q, $\mathbf{E}$, and $\mathbf{J}$ are all functions of time; Ampère and Biot-Savart do not apply. The displacement current

$$J_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{4\pi} \frac{\dot{Q}}{r^2}\,\hat{\mathbf{r}} = -\sigma \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}$$

exactly cancels the conduction current (in Eq. 7.37), and the magnetic field (determined by $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{B} = \mathbf{0}$) is indeed zero.

Problem 7.34 A fat wire, radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor, as shown in Fig. 7.45. Find the magnetic field in the gap, at a distance $s < a$ from the axis.

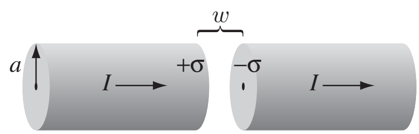


FIGURE 7.45

Problem 7.35 The preceding problem was an artificial model for the charging capacitor, designed to avoid complications associated with the current spreading out over the surface of the plates. For a more realistic model, imagine *thin* wires that connect to the centers of the plates (Fig. 7.46a). Again, the current I is constant, the radius of the capacitor is a, and the separation of the plates is $w \ll a$. Assume that the current flows out over the plates in such a way that the surface charge is uniform, at any given time, and is zero at $t = 0$.

(a) Find the electric field between the plates, as a function of t.

(b) Find the displacement current through a circle of radius s in the plane midway between the plates. Using this circle as your "Amperian loop," and the flat surface that spans it, find the magnetic field at a distance s from the axis.

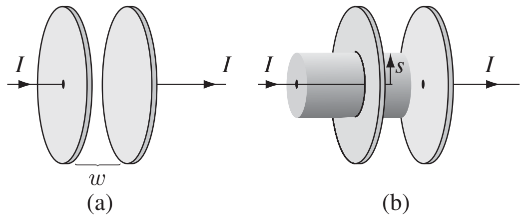


FIGURE 7.46

(c) Repeat part (b), but this time use the cylindrical surface in Fig. 7.46(b), which is open at the right end and extends to the left through the plate and terminates outside the capacitor. Notice that the displacement current through this surface is zero, and there are two contributions to I_{enc}.[22]

Problem 7.36 Refer to Prob. 7.16, to which the correct answer was

$$\mathbf{E}(s, t) = \frac{\mu_0 I_0 \omega}{2\pi} \sin(\omega t) \ln\left(\frac{a}{s}\right) \hat{\mathbf{z}}.$$

(a) Find the displacement current density $\mathbf{J}_d$.

(b) Integrate it to get the total displacement current,

$$I_d = \int \mathbf{J}_d \cdot d\mathbf{a}.$$

(c) Compare I_d and I. (What's their ratio?) If the outer cylinder were, say, 2 mm in diameter, how high would the frequency have to be, for I_d to be 1% of I? [This problem is designed to indicate why Faraday never discovered displacement currents, and why it is ordinarily safe to ignore them unless the frequency is extremely high.]

7.3.3 ■ Maxwell's Equations

In the last section we put the finishing touches on Maxwell's equations:

$$\left.\begin{array}{llll}
\text{(i)} & \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho & & \text{(Gauss's law),} \\
\text{(ii)} & \nabla \cdot \mathbf{B} = 0 & & \text{(no name),} \\
\text{(iii)} & \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} & & \text{(Faraday's law),} \\
\text{(iv)} & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} & & \text{(Ampère's law with Maxwell's correction).}
\end{array}\right. \tag{7.40}$$

[22]This problem raises an interesting quasi-philosophical question: If you measure **B** in the laboratory, have you detected the effects of displacement current (as (b) would suggest), or merely confirmed the effects of ordinary currents (as (c) implies)? See D. F. Bartlett, *Am. J. Phys.* **58**, 1168 (1990).

Together with the force law,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{7.41}$$

they summarize the entire theoretical content of classical electrodynamics[23] (save for some special properties of matter, which we encountered in Chapters 4 and 6). Even the continuity equation,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}, \tag{7.42}$$

which is the mathematical expression of conservation of charge, can be derived from Maxwell's equations by applying the divergence to number (iv).

I have written Maxwell's equations in the traditional way, which emphasizes that they specify the divergence and curl of **E** and **B**. In this form, they reinforce the notion that electric fields can be produced *either* by charges (ρ) *or* by changing magnetic fields ($\partial \mathbf{B}/\partial t$), and magnetic fields can be produced *either* by currents (**J**) *or* by changing electric fields ($\partial \mathbf{E}/\partial t$). Actually, this is misleading, because $\partial \mathbf{B}/\partial t$ and $\partial \mathbf{E}/\partial t$ are *themselves* due to charges and currents. I think it is logically preferable to write

$$\left.\begin{array}{ll} \text{(i)}\ \ \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(iii)}\ \ \nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \\ \text{(ii)}\ \ \nabla \cdot \mathbf{B} = 0, & \text{(iv)}\ \ \nabla \times \mathbf{B} - \mu_0\epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}, \end{array}\right\} \tag{7.43}$$

with the fields (**E** and **B**) on the left and the sources (ρ and **J**) on the right. This notation emphasizes that all electromagnetic fields are ultimately attributable to charges and currents. Maxwell's equations tell you how *charges* produce *fields*; reciprocally, the force law tells you how *fields* affect *charges*.

Problem 7.37 Suppose

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt - r)\hat{\mathbf{r}}; \quad \mathbf{B}(\mathbf{r}, t) = \mathbf{0}$$

(The theta function is defined in Prob. 1.46b). Show that these fields satisfy all of Maxwell's equations, and determine ρ and **J**. Describe the physical situation that gives rise to these fields.

7.3.5 ■ Maxwell's Equations in Matter

Maxwell's equations in the form 7.40 are complete and correct as they stand. However, when you are working with materials that are subject to electric and magnetic polarization there is a more convenient way to *write* them. For inside polarized matter there will be accumulations of "bound" charge and current, over which you exert no direct control. It would be nice to reformulate Maxwell's equations so as to make explicit reference only to the "free" charges and currents.

We have already learned, from the static case, that an electric polarization $\mathbf{P}$ produces a bound charge density

$$\rho_b = -\nabla \cdot \mathbf{P} \tag{7.47}$$

(Eq. 4.12). Likewise, a magnetic polarization (or "magnetization") $\mathbf{M}$ results in a bound current

$$\mathbf{J}_b = \nabla \times \mathbf{M} \tag{7.48}$$

(Eq. 6.13). There's just one new feature to consider in the *non*static case: Any *change* in the electric polarization involves a flow of (bound) charge (call it $\mathbf{J}_p$), which must be included in the total current. For suppose we examine a tiny chunk of polarized material (Fig. 7.47). The polarization introduces a charge density $\sigma_b = P$ at one end and $-\sigma_b$ at the other (Eq. 4.11). If P now *increases* a bit, the charge on each end increases accordingly, giving a net current

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp.$$

The current density, therefore, is

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}. \tag{7.49}$$

This **polarization current** has nothing to do with the *bound* current $\mathbf{J}_b$. The latter is associated with *magnetization* of the material and involves the spin and orbital motion of electrons; $\mathbf{J}_p$, by contrast, is the result of the linear motion of charge when the electric polarization changes. If $\mathbf{P}$ points to the right, and is increasing, then each plus charge moves a bit to the right and each minus charge to the left; the cumulative effect is the polarization current $\mathbf{J}_p$. We ought to check that Eq. 7.49 is consistent with the continuity equation:

$$\nabla \cdot \mathbf{J}_p = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{P}) = -\frac{\partial \rho_b}{\partial t}.$$

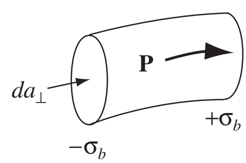


FIGURE 7.47

Yes: The continuity equation *is* satisfied; in fact, $\mathbf{J}_p$ is essential to ensure the conservation of bound charge. (Incidentally, a changing *magnetization* does *not* lead to any analogous accumulation of charge or current. The bound current $\mathbf{J}_b = \nabla \times \mathbf{M}$ varies in response to changes in $\mathbf{M}$, to be sure, but that's about it.)

In view of all this, the total charge density can be separated into two parts:

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}, \tag{7.50}$$

and the current density into *three* parts:

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}. \tag{7.51}$$

Gauss's law can now be written as

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho_f - \nabla \cdot \mathbf{P}),$$

or

$$\nabla \cdot \mathbf{D} = \rho_f, \tag{7.52}$$

where, as in the static case,

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}. \tag{7.53}$$

Meanwhile, Ampère's law (with Maxwell's term) becomes

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

or

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \tag{7.54}$$

where, as before,

$$\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}. \tag{7.55}$$

Faraday's law and $\nabla \cdot \mathbf{B} = 0$ are not affected by our separation of charge and current into free and bound parts, since they do not involve ρ or $\mathbf{J}$.

In terms of *free* charges and currents, then, Maxwell's equations read

$$\begin{array}{|ll|} \hline \text{(i)}\ \nabla \cdot \mathbf{D} = \rho_f, & \text{(iii)}\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)}\ \nabla \cdot \mathbf{B} = 0, & \text{(iv)}\ \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}. \\ \hline \end{array} \tag{7.56}$$

Some people regard these as the "true" Maxwell's equations, but please understand that they are in *no way* more "general" than Eq. 7.40; they simply reflect a convenient division of charge and current into free and nonfree parts. And they have the disadvantage of hybrid notation, since they contain both **E** and **D**, both **B** and **H**. They must be supplemented, therefore, by appropriate **constitutive relations**, giving **D** and **H** in terms of **E** and **B**. These depend on the nature of the material; for linear media

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \text{and} \quad \mathbf{M} = \chi_m \mathbf{H}, \tag{7.57}$$

so

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}, \tag{7.58}$$

where $\epsilon \equiv \epsilon_0(1 + \chi_e)$ and $\mu \equiv \mu_0(1 + \chi_m)$. Incidentally, you'll remember that **D** is called the electric "displacement"; that's why the second term in the Ampère/Maxwell equation (iv) came to be called the **displacement current**. In this context,

$$\mathbf{J}_d \equiv \frac{\partial \mathbf{D}}{\partial t}. \tag{7.59}$$

Problem 7.40 Sea water at frequency $\nu = 4 \times 10^8$ Hz has permittivity $\epsilon = 81\epsilon_0$, permeability $\mu = \mu_0$, and resistivity $\rho = 0.23\ \Omega \cdot \text{m}$. What is the ratio of conduction current to displacement current? [*Hint:* Consider a parallel-plate capacitor immersed in sea water and driven by a voltage $V_0 \cos(2\pi \nu t)$.]

7.3.6 ■ Boundary Conditions

In general, the fields **E, B, D,** and **H** will be discontinuous at a boundary between two different media, or at a surface that carries a charge density σ or a current density **K**. The explicit form of these discontinuities can be deduced from Maxwell's equations (7.56), in their integral form

$$\left.\begin{array}{ll}\text{(i)} & \oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}} \\ \text{(ii)} & \oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = 0\end{array}\right\} \text{over any closed surface } \mathcal{S}.$$

$$\left.\begin{array}{ll}\text{(iii)} & \oint_{\mathcal{P}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} \\ \text{(iv)} & \oint_{\mathcal{P}} \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} + \frac{d}{dt}\int_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a}\end{array}\right\} \begin{array}{l}\text{for any surface } \mathcal{S} \\ \text{bounded by the} \\ \text{closed loop } \mathcal{P}.\end{array}$$

Applying (i) to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary (Fig. 7.48), we obtain:

$$\mathbf{D}_1 \cdot \mathbf{a} - \mathbf{D}_2 \cdot \mathbf{a} = \sigma_f\, a.$$

(The positive direction for **a** is *from* 2 *toward* 1. The edge of the wafer contributes nothing in the limit as the thickness goes to zero; nor does any *volume*

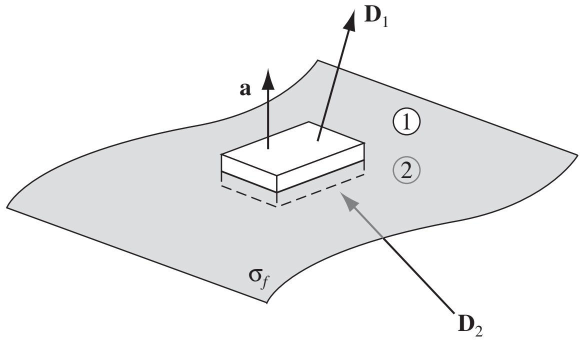


FIGURE 7.48

charge density.) Thus, the component of **D** that is perpendicular to the interface is discontinuous in the amount

$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f.} \tag{7.60}$$

Identical reasoning, applied to equation (ii), yields

$$\boxed{B_1^{\perp} - B_2^{\perp} = 0.} \tag{7.61}$$

Turning to (iii), a very thin Amperian loop straddling the surface gives

$$\mathbf{E}_1 \cdot \mathbf{l} - \mathbf{E}_2 \cdot \mathbf{l} = -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a}.$$

But in the limit as the width of the loop goes to zero, the flux vanishes. (I have already dropped the contribution of the two ends to $\oint \mathbf{E} \cdot d\mathbf{l}$, on the same grounds.) Therefore,

$$\boxed{\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}.} \tag{7.62}$$

That is, the components of **E** *parallel* to the interface are continuous across the boundary. By the same token, (iv) implies

$$\mathbf{H}_1 \cdot \mathbf{l} - \mathbf{H}_2 \cdot \mathbf{l} = I_{f_{\text{enc}}},$$

where $I_{f_{\text{enc}}}$ is the free current passing through the Amperian loop. No *volume* current density will contribute (in the limit of infinitesimal width), but a *surface* current can. In fact, if $\hat{\mathbf{n}}$ is a unit vector perpendicular to the interface (pointing from 2 toward 1), so that $(\hat{\mathbf{n}} \times \mathbf{l})$ is normal to the Amperian loop (Fig. 7.49), then

$$I_{f_{\text{enc}}} = \mathbf{K}_f \cdot (\hat{\mathbf{n}} \times \mathbf{l}) = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \mathbf{l},$$

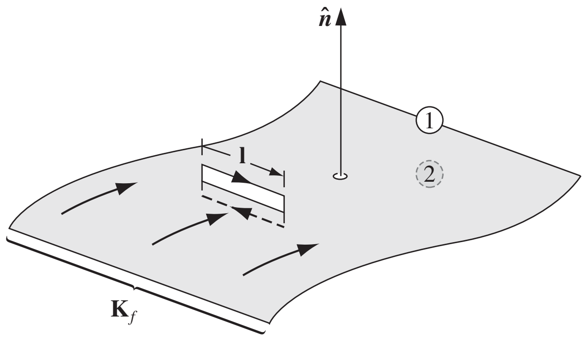


FIGURE 7.49

and hence

$$\boxed{\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.} \tag{7.63}$$

So the *parallel* components of **H** are discontinuous by an amount proportional to the free surface current density.

Equations 7.60-63 are the general boundary conditions for electrodynamics. In the case of *linear* media, they can be expressed in terms of **E** and **B** alone:

$$\left.\begin{array}{ll} \text{(i)}\ \ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, & \text{(iii)}\ \ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}, \\ \text{(ii)}\ \ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}. \end{array}\right\} \tag{7.64}$$

In particular, if there is no free charge or free current at the interface, then

$$\begin{array}{ll} \text{(i)}\ \ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0, & \text{(iii)}\ \ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}, \\ \text{(ii)}\ \ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{0}. \end{array} \tag{7.65}$$

As we shall see in Chapter 9, these equations are the basis for the theory of reflection and refraction.

More Problems on Chapter 7

! **Problem 7.41** Two long, straight copper pipes, each of radius a, are held a distance $2d$ apart (see Fig. 7.50). One is at potential V_0, the other at $-V_0$. The space surrounding the pipes is filled with weakly conducting material of conductivity σ. Find the current per unit length that flows from one pipe to the other. [*Hint:* Refer to Prob. 3.12.]

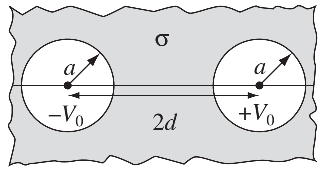


FIGURE 7.50

! **Problem 7.42** A rare case in which the electrostatic field **E** for a circuit can actually be *calculated* is the following:[28] Imagine an infinitely long cylindrical sheet, of uniform resistivity and radius a. A slot (corresponding to the battery) is maintained at $\pm V_0/2$, at $\phi = \pm\pi$, and a steady current flows over the surface, as indicated in Fig. 7.51. According to Ohm's law, then,

$$V(a, \phi) = \frac{V_0 \phi}{2\pi}, \quad (-\pi < \phi < +\pi).$$

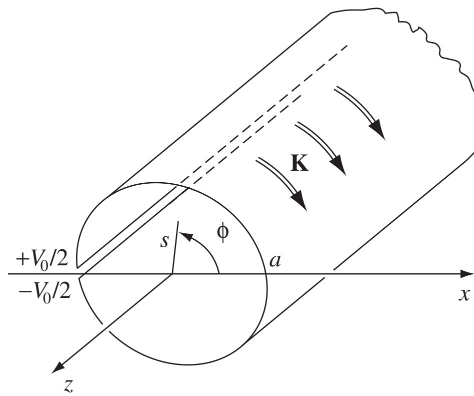


FIGURE 7.51

(a) Use separation of variables in cylindrical coordinates to determine $V(s, \phi)$ inside and outside the cylinder. [*Answer:* $(V_0/\pi)\tan^{-1}[(s\sin\phi)/(a + s\cos\phi)]$, $(s < a)$; $(V_0/\pi)\tan^{-1}[(a\sin\phi)/(s + a\cos\phi)]$, $(s > a)$]

(b) Find the surface charge density on the cylinder. [*Answer:* $(\epsilon_0 V_0/\pi a)\tan(\phi/2)$]

Problem 7.43 The magnetic field outside a long straight wire carrying a steady current I is

$$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\,\hat{\boldsymbol{\phi}}.$$

The *electric field inside* the wire is uniform:

$$\mathbf{E} = \frac{I\rho}{\pi a^2}\,\hat{\mathbf{z}},$$

[28] M. A. Heald, *Am. J. Phys.* **52**, 522 (1984). See also J. A. Hernandes and A. K. T. Assis, *Phys. Rev. E* **68**, 046611 (2003).

where ρ is the resistivity and a is the radius (see Exs. 7.1 and 7.3). *Question:* What is the electric field *outside* the wire?[29] The answer depends on how you complete the circuit. Suppose the current returns along a perfectly conducting grounded coaxial cylinder of radius b (Fig. 7.52). In the region $a < s < b$, the potential $V(s, z)$ satisfies Laplace's equation, with the boundary conditions

$$\text{(i)}\ \ V(a, z) = -\frac{I\rho z}{\pi a^2}; \quad \text{(ii)}\ \ V(b, z) = 0.$$

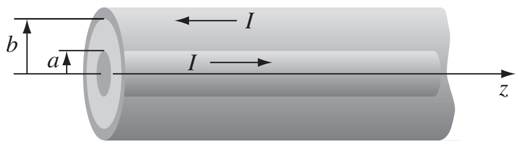


FIGURE 7.52

This does not suffice to determine the answer—we still need to specify boundary conditions at the two ends (though for a *long* wire it shouldn't matter much). In the literature, it is customary to sweep this ambiguity under the rug by simply *stipulating* that $V(s, z)$ is proportional to z: $V(s, z) = zf(s)$. On this assumption:

(a) Determine $f(s)$.

(b) Find $\mathbf{E}(s, z)$.

(c) Calculate the surface charge density $\sigma(z)$ on the wire.

[*Answer:* $V = (-Iz\rho/\pi a^2)[\ln(s/b)/\ln(a/b)]$ This is a peculiar result, since E_s and $\sigma(z)$ are *not* independent of z—as one would certainly expect for a truly *infinite* wire.]

Problem 7.44 In a **perfect conductor**, the conductivity is infinite, so $\mathbf{E} = \mathbf{0}$ (Eq. 7.3), and any net charge resides on the surface (just as it does for an *im*perfect conductor, in electro*statics*).

(a) Show that the magnetic field is constant ($\partial\mathbf{B}/\partial t = \mathbf{0}$), inside a perfect conductor.

(b) Show that the magnetic flux through a perfectly conducting loop is constant.

A **superconductor** is a perfect conductor with the additional property that the (constant) $\mathbf{B}$ inside is in fact *zero*. (This "flux exclusion" is known as the **Meissner effect**.[30])

[29] This is a famous problem, first analyzed by Sommerfeld, and is known in its most recent incarnation as **Merzbacher's puzzle**. A. Sommerfeld, *Electrodynamics*, p. 125 (New York: Academic Press, 1952); E. Merzbacher, *Am. J. Phys.* **48**, 178 (1980); further references in R. N. Varnay and L. H. Fisher, *Am. J. Phys.* **52**, 1097 (1984).

[30] The Meissner effect is sometimes referred to as "perfect diamagnetism," in the sense that the field inside is not merely *reduced*, but canceled entirely. However, the surface currents responsible for this are *free*, not bound, so the actual *mechanism* is quite different.

(c) Show that the current in a superconductor is confined to the surface.

(d) Superconductivity is lost above a certain critical temperature (T_c), which varies from one material to another. Suppose you had a sphere (radius a) above its critical temperature, and you held it in a uniform magnetic field $B_0\hat{\mathbf{z}}$ while cooling it below T_c. Find the induced surface current density **K**, as a function of the polar angle θ.

Problem 7.45 A familiar demonstration of superconductivity (Prob. 7.44) is the levitation of a magnet over a piece of superconducting material. This phenomenon can be analyzed using the method of images.[31] Treat the magnet as a perfect dipole **m**, a height z above the origin (and constrained to point in the z direction), and pretend that the superconductor occupies the entire half-space below the xy plane. Because of the Meissner effect, $\mathbf{B} = \mathbf{0}$ for $z \leq 0$, and since **B** is divergenceless, the normal (z) component is continuous, so $B_z = 0$ just *above* the surface. This boundary condition is met by the image configuration in which an identical dipole is placed at $-z$, as a stand-in for the superconductor; the two arrangements therefore produce the same magnetic field in the region $z > 0$.

(a) Which way should the image dipole point ($+z$ or $-z$)?

(b) Find the force on the magnet due to the induced currents in the superconductor (which is to say, the force due to the image dipole). Set it equal to Mg (where M is the mass of the magnet) to determine the height h at which the magnet will "float." [*Hint:* Refer to Prob. 6.3.]

(c) The induced current on the surface of the superconductor (the xy plane) can be determined from the boundary condition on the *tangential* component of **B** (Eq. 5.76): $\mathbf{B} = \mu_0(\mathbf{K} \times \hat{\mathbf{z}})$. Using the field you get from the image configuration, show that

$$\mathbf{K} = -\frac{3mrh}{2\pi(r^2 + h^2)^{5/2}}\,\hat{\boldsymbol{\phi}},$$

where r is the distance from the origin.

! **Problem 7.46** If a magnetic dipole levitating above an infinite superconducting plane (Prob. 7.45) is free to rotate, what orientation will it adopt, and how high above the surface will it float?

Problem 7.47 A perfectly conducting spherical shell of radius a rotates about the z axis with angular velocity ω, in a uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$. Calculate the emf developed between the "north pole" and the equator. [*Answer:* $\frac{1}{2}B_0\omega a^2$]

! **Problem 7.48** Refer to Prob. 7.11 (and use the result of Prob. 5.42): How long does is take a falling *circular* ring (radius a, mass m, resistance R) to cross the bottom of the magnetic field B, at its (changing) terminal velocity?

[31] W. M. Saslow, *Am. J. Phys.* **59**, 16 (1991).

Problem 7.49

(a) Referring to Prob. 5.52(a) and Eq. 7.18, show that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \tag{7.66}$$

for Faraday-induced electric fields. Check this result by taking the divergence and curl of both sides.

(b) A spherical shell of radius R carries a uniform surface charge σ. It spins about a fixed axis at an angular velocity $\omega(t)$ that changes slowly with time. Find the electric field inside and outside the sphere. [*Hint:* There are *two* contributions here: the Coulomb field due to the charge, and the Faraday field due to the changing **B**. Refer to Ex. 5.11.]

Problem 7.50 Electrons undergoing cyclotron motion can be sped up by increasing the magnetic field; the accompanying electric field will impart tangential acceleration. This is the principle of the **betatron**. One would like to keep the radius of the orbit constant during the process. Show that this can be achieved by designing a magnet such that the average field over the area of the orbit is twice the field at the circumference (Fig. 7.53). Assume the electrons start from rest in zero field, and that the apparatus is symmetric about the center of the orbit. (Assume also that the electron velocity remains well below the speed of light, so that nonrelativistic mechanics applies.) [*Hint:* Differentiate Eq. 5.3 with respect to time, and use $F = ma = qE$.]

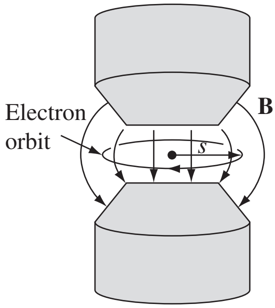


FIGURE 7.53

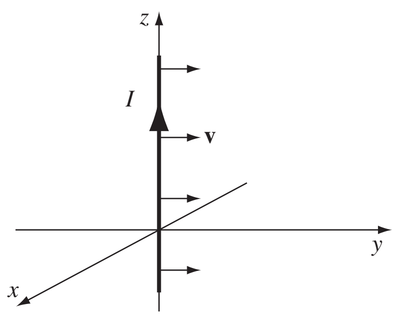


FIGURE 7.54

Problem 7.51 An infinite wire carrying a constant current I in the $\hat{\mathbf{z}}$ direction is moving in the y direction at a constant speed v. Find the electric field, in the quasistatic approximation, at the instant the wire coincides with the z axis (Fig. 7.54). [*Answer:* $-(\mu_0 I v/2\pi s)\sin\phi\,\hat{\mathbf{z}}$]

Problem 7.52 An atomic electron (charge q) circles about the nucleus (charge Q) in an orbit of radius r; the centripetal acceleration is provided, of course, by the Coulomb attraction of opposite charges. Now a small magnetic field dB is slowly turned on, perpendicular to the plane of the orbit. Show that the increase in kinetic energy, dT, imparted by the induced electric field, is just right to sustain circular motion *at the same radius* r. (That's why, in my discussion of diamagnetism, I assumed the radius is fixed. See Sect. 6.1.3 and the references cited there.)

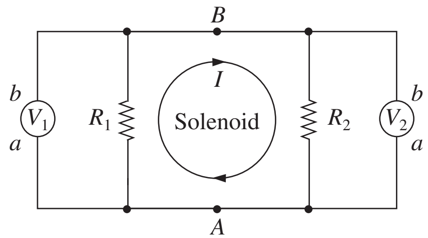


FIGURE 7.55

Problem 7.53 The current in a long solenoid is increasing linearly with time, so the flux is proportional to t: $\Phi = \alpha t$. Two voltmeters are connected to diametrically opposite points (A and B), together with resistors (R_1 and R_2), as shown in Fig. 7.55. What is the reading on each voltmeter? Assume that these are *ideal* voltmeters that draw negligible current (they have huge internal resistance), and that a voltmeter registers $-\int_a^b \mathbf{E}\cdot d\mathbf{l}$ between the terminals and through the meter. [*Answer:* $V_1 = \alpha R_1/(R_1 + R_2)$; $V_2 = -\alpha R_2/(R_1 + R_2)$. Notice that $V_1 \neq V_2$, even though they are connected to the same points![32]]

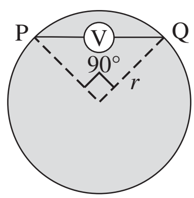


FIGURE 7.56

Problem 7.54 A circular wire loop (radius r, resistance R) encloses a region of uniform magnetic field, B, perpendicular to its plane. The field (occupying the shaded region in Fig. 7.56) increases linearly with time ($B = \alpha t$). An ideal voltmeter (infinite internal resistance) is connected between points P and Q.

(a) What is the current in the loop?

(b) What does the voltmeter read? [*Answer:* $\alpha r^2/2$]

Problem 7.55 In the discussion of motional emf (Sect. 7.1.3) I assumed that the wire loop (Fig. 7.10) has a resistance R; the current generated is then $I = vBh/R$. But what if the wire is made out of perfectly conducting material, so that R is *zero*? In that case, the current is limited only by the back emf associated with the self-inductance L of the loop (which would ordinarily be negligible in comparison with IR). Show that in this régime the loop (mass m) executes simple harmonic motion, and find its frequency.[33] [*Answer:* $\omega = Bh/\sqrt{mL}$]

[32] R. H. Romer, *Am. J. Phys.* **50**, 1089 (1982). See also H. W. Nicholson, *Am. J. Phys.* **73**, 1194 (2005); B. M. McGuyer, *Am. J. Phys.* **80**, 101 (2012).

[33] For a collection of related problems, see W. M. Saslow, *Am. J. Phys.* **55**, 986 (1987), and R. H. Romer, *Eur. J. Phys.* **11**, 103 (1990).

Problem 7.56

(a) Use the Neumann formula (Eq. 7.23) to calculate the mutual inductance of the configuration in Fig. 7.37, assuming a is very small ($a \ll b, a \ll z$). Compare your answer to Prob. 7.22.

(b) For the general case (*not* assuming a is small), show that

$$M = \frac{\mu_0 \pi \beta}{2} \sqrt{ab\beta} \left(1 + \frac{15}{8}\beta^2 + \dots\right),$$

where

$$\beta \equiv \frac{ab}{z^2 + a^2 + b^2}.$$

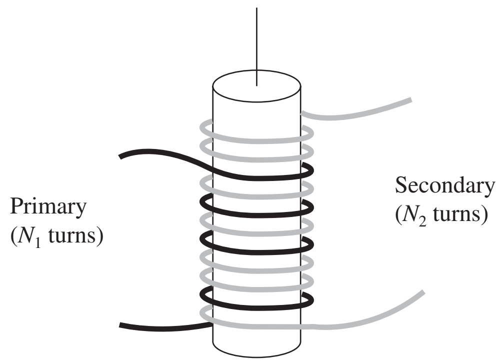


FIGURE 7.57

Problem 7.57 Two coils are wrapped around a cylindrical form in such a way that the *same flux passes through every turn of both coils*. (In practice this is achieved by inserting an iron core through the cylinder; this has the effect of concentrating the flux.) The **primary** coil has N_1 turns and the **secondary** has N_2 (Fig. 7.57). If the current I in the primary is changing, show that the emf in the secondary is given by

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}, \tag{7.67}$$

where $\mathcal{E}_1$ is the (back) emf of the primary. [This is a primitive **transformer**—a device for raising or lowering the emf of an alternating current source. By choosing the appropriate number of turns, any desired secondary emf can be obtained. If you think this violates the conservation of energy, study Prob. 7.58.]

Problem 7.58 A transformer (Prob. 7.57) takes an input AC voltage of amplitude V_1, and delivers an output voltage of amplitude V_2, which is determined by the turns ratio ($V_2/V_1 = N_2/N_1$). If $N_2 > N_1$, the output voltage is greater than the input voltage. Why doesn't this violate conservation of energy? *Answer:* Power is the product of voltage and current; if the voltage goes *up*, the current must come *down*. The purpose of this problem is to see exactly how this works out, in a simplified model.

(a) In an ideal transformer, the same flux passes through all turns of the primary and of the secondary. Show that in this case $M^2 = L_1 L_2$, where M is the mutual inductance of the coils, and L_1, L_2 are their individual self-inductances.

(b) Suppose the primary is driven with AC voltage $V_{\text{in}} = V_1 \cos(\omega t)$, and the secondary is connected to a resistor, R. Show that the two currents satisfy the relations

$$L_1 \frac{dI_1}{dt} + M\frac{dI_2}{dt} = V_1 \cos(\omega t); \quad L_2 \frac{dI_2}{dt} + M\frac{dI_1}{dt} = -I_2 R.$$

(c) Using the result in (a), solve these equations for $I_1(t)$ and $I_2(t)$. (Assume I_1 has no DC component.)

(d) Show that the output voltage ($V_{\text{out}} = I_2 R$) divided by the input voltage (V_{in}) is equal to the turns ratio: $V_{\text{out}}/V_{\text{in}} = N_2/N_1$.

(e) Calculate the input power ($P_{\text{in}} = V_{\text{in}} I_1$) and the output power ($P_{\text{out}} = V_{\text{out}} I_2$), and show that their averages over a full cycle are equal.

Problem 7.59 An infinite wire runs along the z axis; it carries a current $I(z)$ that is a function of z (but not of t), and a charge density $\lambda(t)$ that is a function of t (but not of z).

(a) By examining the charge flowing into a segment dz in a time dt, show that $d\lambda/dt = -dI/dz$. If we stipulate that $\lambda(0) = 0$ and $I(0) = 0$, show that $\lambda(t) = kt,\ I(z) = -kz$, where k is a constant.

(b) Assume for a moment that the process is quasistatic, so the fields are given by Eqs. 2.9 and 5.38. Show that these are in fact the *exact* fields, by confirming that all four of Maxwell's equations are satisfied. (First do it in differential form, for the region $s > 0$, then in integral form for the appropriate Gaussian cylinder/Amperian loop straddling the axis.)

Problem 7.60 Suppose $\mathbf{J}(\mathbf{r})$ is constant in time but $\rho(\mathbf{r}, t)$ is *not*—conditions that might prevail, for instance, during the charging of a capacitor.

(a) Show that the charge density at any particular point is a linear function of time:

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0) + \dot{\rho}(\mathbf{r}, 0)t,$$

where $\dot{\rho}(\mathbf{r}, 0)$ is the time derivative of ρ at $t = 0$. [*Hint:* Use the continuity equation.]

This is *not* an electrostatic or magnetostatic configuration;[34] nevertheless, rather surprisingly, both Coulomb's law (Eq. 2.8) and the Biot-Savart law (Eq. 5.42) hold, as you can confirm by showing that they satisfy Maxwell's equations. In particular:

[34]Some authors *would* regard this as magnetostatic, since $\mathbf{B}$ is independent of t. For them, the Biot-Savart law is a general rule of magnetostatics, but $\nabla \cdot \mathbf{J} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ apply only under the *additional* assumption that ρ is constant. In such a formulation, Maxwell's displacement term can (in this very special case) be *derived* from the Biot-Savart law, by the method of part (b). See D. F. Bartlett, *Am. J. Phys.* **58**, 1168 (1990); D. J. Griffiths and M. A. Heald, *Am. J. Phys.* **59**, 111 (1991).

(b) Show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$

obeys Ampère's law *with Maxwell's displacement current term.*

Problem 7.61 The magnetic field of an infinite straight wire carrying a steady current I can be obtained from the *displacement* current term in the Ampère/Maxwell law, as follows: Picture the current as consisting of a uniform line charge λ moving along the z axis at speed v (so that $I = \lambda v$), with a tiny gap of length ϵ, which reaches the origin at time $t = 0$. In the next instant (up to $t = \epsilon/v$) there is no *real* current passing through a circular Amperian loop in the xy plane, but there *is* a *displacement* current, due to the "missing" charge in the gap.

(a) Use Coulomb's law to calculate the z component of the electric field, for points in the xy plane a distance s from the origin, due to a segment of wire with uniform density $-\lambda$ extending from $z_1 = vt - \epsilon$ to $z_2 = vt$.

(b) Determine the flux of this electric field through a circle of radius a in the xy plane.

(c) Find the displacement current through this circle. Show that I_d is equal to I, in the limit as the gap width (ϵ) goes to zero.[35]

Problem 7.62 A certain transmission line is constructed from two thin metal "ribbons," of width w, a very small distance $h \ll w$ apart. The current travels down one strip and back along the other. In each case, it spreads out uniformly over the surface of the ribbon.

(a) Find the capacitance per unit length, $\mathcal{C}$.

(b) Find the inductance per unit length, $\mathcal{L}$.

(c) What is the product $\mathcal{LC}$, numerically? [$\mathcal{L}$ and $\mathcal{C}$ will, of course, vary from one kind of transmission line to another, but their *product* is a universal constant—check, for example, the cable in Ex. 7.13—provided the space between the conductors is a vacuum. In the theory of transmission lines, this product is related to the speed with which a pulse propagates down the line: $v = 1/\sqrt{\mathcal{LC}}$.]

(d) If the strips are insulated from one another by a nonconducting material of permittivity ϵ and permeability μ, what then is the product $\mathcal{LC}$? What is the propagation speed? [*Hint:* see Ex. 4.6; by what factor does L change when an inductor is immersed in linear material of permeability μ?]

Problem 7.63 Prove **Alfven's theorem**: In a perfectly conducting fluid (say, a gas of free electrons), the magnetic flux through any closed loop moving with the fluid is constant in time. (The magnetic field lines are, as it were, "frozen" into the fluid.)

(a) Use Ohm's law, in the form of Eq. 7.2, together with Faraday's law, to prove that if $\sigma = \infty$ and $\mathbf{J}$ is finite, then

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).$$

[35]For a slightly different approach to the same problem, see W. K. Terry, *Am. J. Phys.* **50**, 742 (1982).

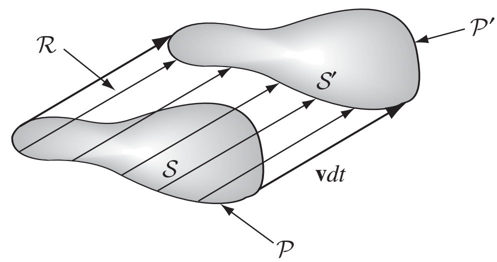


FIGURE 7.58

(b) Let $\mathcal{S}$ be the surface bounded by the loop ($\mathcal{P}$) at time t, and $\mathcal{S}'$ a surface bounded by the loop in its new position ($\mathcal{P}'$) at time $t + dt$ (see Fig. 7.58). The change in flux is

$$d\Phi = \int_{\mathcal{S}'} \mathbf{B}(t + dt) \cdot d\mathbf{a} - \int_{\mathcal{S}} \mathbf{B}(t) \cdot d\mathbf{a}.$$

Use $\nabla \cdot \mathbf{B} = 0$ to show that

$$\int_{\mathcal{S}'} \mathbf{B}(t + dt) \cdot d\mathbf{a} + \int_{\mathcal{R}} \mathbf{B}(t + dt) \cdot d\mathbf{a} = \int_{\mathcal{S}} \mathbf{B}(t + dt) \cdot d\mathbf{a}$$

(where $\mathcal{R}$ is the "ribbon" joining $\mathcal{P}$ and $\mathcal{P}'$), and hence that

$$d\Phi = dt \int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} - \int_{\mathcal{R}} \mathbf{B}(t + dt) \cdot d\mathbf{a}$$

(for infinitesimal dt). Use the method of Sect. 7.1.3 to rewrite the second integral as

$$dt \oint_{\mathcal{P}} (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l},$$

and invoke Stokes' theorem to conclude that

$$\frac{d\Phi}{dt} = \int_{\mathcal{S}} \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) \right) \cdot d\mathbf{a}.$$

Together with the result in (a), this proves the theorem.

Problem 7.64

(a) Show that Maxwell's equations with magnetic charge (Eq. 7.44) are invariant under the **duality transformation**

$$\left.\begin{aligned} \mathbf{E}' &= \mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha, \\ c\mathbf{B}' &= c\mathbf{B}\cos\alpha - \mathbf{E}\sin\alpha, \\ cq_e' &= cq_e\cos\alpha + q_m\sin\alpha, \\ q_m' &= q_m\cos\alpha - cq_e\sin\alpha, \end{aligned}\right\} \tag{7.68}$$

where $c \equiv 1/\sqrt{\epsilon_0\mu_0}$ and α is an arbitrary rotation angle in "**E**/**B**-space." Charge and current densities transform in the same way as q_e and q_m. [This means, in particular, that if you know the fields produced by a configuration of *electric* charge, you can immediately (using $\alpha = 90°$) write down the fields produced by the corresponding arrangement of *magnetic* charge.]

(b) Show that the force law (Prob. 7.38)

$$\mathbf{F} = q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + q_m\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) \tag{7.69}$$

is also invariant under the duality transformation.

9.2 ■ ELECTROMAGNETIC WAVES IN VACUUM

9.2.1 ■ The Wave Equation for E and B

In regions of space where there is no charge or current, Maxwell's equations read

$$\left.\begin{aligned} &\text{(i)} \quad \nabla \cdot \mathbf{E} = 0, \quad \text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ &\text{(ii)} \quad \nabla \cdot \mathbf{B} = 0, \quad \text{(iv)} \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned}\right\} \tag{9.40}$$

They constitute a set of coupled, first-order, partial differential equations for **E** and **B**. They can be *de*coupled by applying the curl to (iii) and (iv):

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) \\ &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \end{aligned}$$

[3] An elegant notation for circular polarization (or elliptical, if the amplitudes are unequal) is to use a *complex* $\hat{\mathbf{n}}$, but I shall not do so in this book.

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$= \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

Or, since $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$,

$$\boxed{\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.} \tag{9.41}$$

We now have *separate* equations for $\mathbf{E}$ and $\mathbf{B}$, but they are of *second* order; that's the price you pay for decoupling them.

In vacuum, then, each Cartesian component of $\mathbf{E}$ and $\mathbf{B}$ satisfies the **three-dimensional wave equation**,

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}.$$

(This is the same as Eq. 9.2, except that $\partial^2 f/\partial z^2$ is replaced by its natural generalization, $\nabla^2 f$.) So Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, traveling at a speed

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \text{ m/s}, \tag{9.42}$$

which happens to be precisely the velocity of light, c. The implication is astounding: Perhaps light *is* an electromagnetic wave.[4] Of course, this conclusion does not surprise anyone today, but imagine what a revelation it was in Maxwell's time! Remember how ϵ_0 and μ_0 came into the theory in the first place: they were constants in Coulomb's law and the Biot-Savart law, respectively. You measure them in experiments involving charged pith balls, batteries, and wires—experiments having nothing whatever to do with light. And yet, according to Maxwell's theory, you can calculate c from these two numbers. Notice the crucial role played by Maxwell's contribution to Ampère's law ($\mu_0 \epsilon_0 \partial \mathbf{E}/\partial t$); without it, the wave equation would not emerge, and there would be no electromagnetic theory of light.

■